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ESTIMATION OF GROWING PERTURBATION PARAMETERS IN SHEAR FLOWS OF A VISCOUS STRATIFIED FLUID*

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Estimates of growing linear perturbation parameters in shear plane-parallel viscous fluid flows are obtained, based on the integral relations resulting from the generalized Orr-Sommerfeld equation taking stratification into account and boundaries are determined for the domain containing the complex phase velocity.

1. Attempts to construct a stability theory for shear flows in a linear approximation while simultaneously taking account of the effects of viscosity and stratification were apparently first made by Drazin /1/ who derived the fundamental equation describing the vertical structure of the perturbed stream function for plane-parallel flows. For a fluid with constant viscosity in the Boussinesq approximation /2/ this equation takes the form

$$(U - c)(\varphi'' - \alpha^2\varphi) - U''\varphi + \mathrm{Ri}\, N^2 (U - c)^{-1}\varphi = -i(\alpha\,\mathrm{Re})^{-1} \times (\varphi^{IV} - \alpha^2\varphi'' + \alpha^4\varphi) \qquad (1.1)$$
$$N^2 = -g\,\rho^{-1}d\rho/dz$$

The prime denotes differentiation with respect to the vertical coordinate z, $\varphi(z)$ is the part of the stream function ψ for perturbations of the form $\psi = \varphi(z)\exp(i\alpha(x - ct))$, $U(z)$, $N(z)$ are dimensionless functions describing the Brent-Vaisala velocity and frequency profiles /2/, respectively, governed by the vertical density distribution $\rho(z)$. It is convenient to select the normalization of these functions for the sequel such that their maximum values equal unity. The parameters in (1.1) have the following meaning: Re is the Reynolds number of the main flow, Ri is the Richardson number /2/, and $c = c_r + ic_i$ is a complex phase velocity of the perturbation normalized to the characteristic velocity of the main flow.

Eq.(1.1) supplemented with boundary conditions corresponding to the presence of solid walls at $z = 0$ and $z = 1$

$$\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0 \qquad (1.2)$$

forms a boundary value problem in which c plays the part of the spectral parameter while $\varphi(z)$ is the eigenfunction, where it follows from the form of the perturbed stream function ψ that the presence of a positive imaginary part in c denotes instability for the flow under consideration. Note that (1.1) reduces to the well-known Taylor-Goldstein equation /2/ as $\mathrm{Re} \to \infty$ and to the Orr-Sommerfeld equation for $\mathrm{Ri} = 0$ /3, 4/. If $\mathrm{Ri} = 0$ while $\mathrm{Re} \to \infty$, then (1.1) is the Rayleigh equation /3, 4/.

We turn our attention to the fact that the Drazin Eq.(1.1) remains singular (due to the last component in the left side) even in the presence of viscosity, which when taken into acount ordinarily removes the singularity in the Rayleigh equation and converts it into an Orr-Sommerfeld equation. In this case, the singularity can be eliminated only by taking account of additional physical factors, heat or salt diffusions that influence the density distribution $\rho(z)$ (whereupon the order of Eq.(1.1) is however increased to six).

Thus, determination of the dispersion dependence of c on α for different Re and Ri flow parameters is required in the formulation presented. The values of the parameters that correspond to instability of the fundamental flow $U(z)$ will form a certain domain in the three-dimensional space α, Re, Ri. Investigations conducted earlier were concerned with estimates of the boundaries of this domain in the planes α and Re /3, 4/ and α and Ri /5-8/. The purpose of the present paper is to obtain estimates for the instability domain boundaries in the three-dimensional space of the parameters.

2. We will write the integral resulting from (1.1). To do this we multiply (1.1) by the complex-conjugate function $\bar{\varphi}(z)$ and integrate the result with respect to z between 0 and 1. Taking account of the boundary conditions (1.2), we arrive at a complex integral relation (the limits of integration are omitted for brevity)

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$$i(\alpha \, \mathrm{Re})^{-1}(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)=Q-c(I_1^2+\alpha^2 I_0^2)-\bar{c}\,\mathrm{Ri}\,J \tag{2.1}$$

$$I_0^2=\int|\varphi|^2 dz, \quad I_1^2=\int|\varphi'|^2 dz, \quad I_2^2=\int|\varphi''|^2 dz$$

$$J=\int\frac{N^2|\varphi|^2}{|U-c|^2}dz$$

We separate out real and imaginary part of c in (2.1)

$$c_r=\frac{1}{\Delta_-}\int\left[\left(U\alpha^2+\frac{1}{2}U''-\frac{\mathrm{Ri}\,UN^2}{|U-c|^2}\right)|\varphi|^2+U|\varphi'|^2\right]dz \tag{2.2}$$

$$c_i=\frac{1}{\Delta_+}\left[\frac{1}{2}(Q-\bar{Q})-\frac{1}{\alpha\,\mathrm{Re}}(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)\right] \tag{2.3}$$

$$\Delta_{\pm}=(I_1^2+\alpha^2 I_0^2)\pm\mathrm{Ri}\,J$$

Using these two equations, we obtain constraints on the phase velocity and the perturbation growth increment.

We start with (2.3) by assuming $c_i>0$. Firstly, we note that

$$|Q-\bar{Q}|\leqslant\int|U'(\varphi'\bar{\varphi}-\bar{\varphi}'\varphi)|dz\leqslant 2|U'|_{max}\int|\varphi|\,|\varphi'|dz\leqslant 2|U'|_{max}I_0 I_1$$

The Cauchy-Schwartz inequality is used here. We now estimate the integral J. We have

$$J=N_{max}^2 I_0^2/c_i^2=I_0^2/c_i^2 \tag{2.4}$$

(according to the normalization taken $N_{max}=1$). We afterwards obtain

$$c_i\leqslant K|U'|_{max}-(\alpha\,\mathrm{Re})^{-1}S(\mathrm{Ri},\alpha,c_i), \quad K=I_0 I_1/(I_1^2+\alpha^2 I_0^2),$$
$$S=(\mathrm{Ri},\alpha,c_i)=(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)/(I_1^2+\alpha^2 I_0^2+\mathrm{Ri}\,c_i^{-2}\cdot I_0^2) \tag{2.5}$$

This inequality contains use functionals I_0, I_1, I_2 of the unknown eigenfunction $\varphi(z)$ and its derivatives; consequently, it is not suitable for practical utilization in such form. However, we take account of the obvious inequalities /3/

$$K=\frac{I_0}{I_1}\leqslant\frac{2}{\pi} \quad \text{and} \quad K=\frac{I_0 I_1}{(I_1-\alpha I_0)^2+2\alpha I_0 I_1}\leqslant\frac{1}{2\alpha} \tag{2.6}$$

Furthermore, we find the lower bound of $S(\mathrm{Ri},\alpha,c_i)$. The appropriate variational problem for the extremum of the functional S turns out to be fairly complicated and its solution cannot be found successfully without a computer. An approach associated with searching for approximate lower bounds for S turns out to be more fruitful. It can be shown that in addition to the trivial estimate $S_{min}=0$ it is easy to obtain a number of other estimates that are more meaningful and not equivalent to each other, which will influence the estimation of the desired quantity c_i in the long run.

Different methods to find the lower bound for S are presented in the Appendix. Replacing S by one of the estimates S_m obtained and using (2.6), we strengthen the inequality (2.5)

$$c_i\leqslant\varkappa|U'|_{max}-(\alpha\,\mathrm{Re})^{-1}S_m(\mathrm{Ri},\alpha,c_i), \quad \varkappa=\max(2\pi^{-1},(2\alpha)^{-1}) \tag{2.7}$$

As is seen from (2.7), this inequality yields a domain in parameter space within which the quantity c_i of the instability increment $\gamma=\alpha c_i$, should be enclosed provided there is an instability. The domain boundaries depend on the method of estimating S, but other parameters being equal that estimate should be chosen that yields the domain of minimal dimensions.

We also note the interesting fact that in the presence of stable stratification in the fluid $(\mathrm{Ri}>0, N^2>0)$ the sign of c_i (i.e., perturbation growth or damping) depends explicitly on the Reynolds number but is independent of the Richardson number. Indeed, the denominator in (2.3) for c_i is always positive while the numerator can have different signs depending on the magnitude of the coefficient of Re. Consequently, the sufficient conditions obtained earlier for the stability of a homogeneous fluid /3/ are carried over automatically to a stratified fluid.

We now examine (2.2) and obtain an estimate for the perturbation phase velocity by assuming the fluid to be weakly stratified $(\mathrm{Ri}\ll 1)$. Using the theorem of the mean for values of functions in a segment we write

$$c_r=U(z_1)+{}^1/_2 U''(z_2)I_0^2/\Delta_- \tag{2.8}$$

where z_1, z_2 are certain points within the segment $[0,1]$. The assumption of the smallness of Ri enables us to consider $\Delta_->0$. This will be known to be ensured if the following inequality is satisfied

$$\delta > 0,\ \delta = {}^1/_4\pi^2 + \alpha^2 - \mathrm{Ri}\, c_i^{-2}$$

(this can be shown by using (2.4) and (2.6)).

Let the flow profile be such that $U''(z) \geqslant 0$; then obviously $c_r > U_{min}$, but $c_r \leqslant U_{max} + {}^1/_2 U''_{max}/\delta$ on the other hand. Therefore, in this case we obtain

$$U_{min} \leqslant c_r \leqslant U_{max} + {}^1/_2 U''_{max}/\delta \tag{2.9}$$

If $U''(z) \leqslant 0$ for any z, we similarly find

$$U_{min} + {}^1/_2 U''_{min}/\delta \leqslant c_r \leqslant U_{max} \tag{2.10}$$

Finally, if $U''(z)$ changes sign in the segment $[0, 1]$ then

$$U_{min} + {}^1/_2 U''_{min}/\delta \leqslant c_r \leqslant U_{max} + {}^1/_2 U''_{max}/\delta \tag{2.11}$$

The estimates (2.9)-(2.11) generalize those obtained earlier for a homogeneous fluid /3/. They can be considered to be the same as analogues of the Howard theorem on a semicircle and its generalizations /5-8/ in the sense that they constrain the domain of allowable values of the complex phase velocity of growing perturbations in the complex c plane.

Appendix. We present several different lower bounds for the functional $S(\mathrm{Ri}, \alpha, c_i) \geqslant 0$.

1°. We rewrite S in the form

$$S = \beta^2 + \frac{I_2^2/I_1^2 + \beta^2}{1 + \beta^2 I_0^2/I_1^2} + \frac{\mathrm{Ri}\, c_i^{-4} I_0^2/I_1^2}{1 + \beta^2 I_0^2/I_1^2} - 2\mathrm{Ri}\, c_i^{-2} \tag{A.1}$$
$$\beta^2 = \alpha^2 + \mathrm{Ri}\, c_i^{-2}$$

Furthermore we use the well-known inequalities /3/

$$I_1^2/I_0^2 \geqslant \pi^2/4,\quad I_2^2/I_1^2 \geqslant 4\pi^2,\quad I_2^2/I_0^2 \geqslant (4{,}73)^4 \tag{A.2}$$

and discard the positive fraction proportional to Ri^2 in (A.1). We then obtain

$$S \geqslant \alpha^2 - \frac{\mathrm{Ri}}{c_i^2} + \frac{\pi^2(4\pi^2 + \alpha^2 + \mathrm{Ri}\, c_i^{-2})}{\pi^2 + 4(\alpha^2 + \mathrm{Ri}\, c_i^{-2})} \tag{A.3}$$

2°. Another estimate for S can be obtained by adding the quantity I_2^2 known to be non-negative in the denominator

$$S \geqslant \frac{2\alpha^2(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)}{I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 + \alpha^2(\alpha^2 + 2\mathrm{Ri}\, c_i^{-2})I_0^2} = \frac{2\alpha^2}{1 + \alpha^2(\alpha^2 + 2\mathrm{Ri}\, c_i^{-2})M^{-1}} \tag{A.4}$$
$$M = (I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2)/I_0^2$$

We replace M by a smaller value by using (A.2), strengthening the inequality (A.4) and we finally obtain

$$S \geqslant \frac{2\alpha^2[(4{,}73)^4 + \alpha^2(\pi^2/2 + \alpha^2)]}{(4{,}73)^4 + \alpha^2(\pi^2/2 + 2\alpha^2 + 2\mathrm{Ri}\, c_i^{-2})} \tag{A.5}$$

3°. Still another method of estimation can be obtained for S by discarding the quantity $I_2^2 \geqslant 0$ in the numerator and using the inequality (A.2)

$$S = \frac{\alpha^2(I_1^2 + \alpha^2 I_0^2)}{I_1^2 + \beta^2 I_0^2} + \frac{\alpha^2 I_1^2}{I_1^2 + \beta^2 I_0^2} \geqslant \frac{2\alpha^2(\pi^2 + 2\alpha^2)}{\pi^2 + 4(\alpha^2 + \mathrm{Ri}\, c_i^{-2})} \tag{A.6}$$

A number of lower bounds for the functional S can also be constructed in a similar manner, whereupon different estimates will be obtained for the quantity c_i. Each of the estimates limits a certain domain in the parameter space $(\mathrm{Re}, \mathrm{Ri}, \alpha, c_i)$. The true value of c_i should be within all these domains. In conclusion, we note that the estimates (A.3), (A.5) and (A.6) presented above are independent in the sense that none of them is included in the others uniformly in α.

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THE PASSAGE OF A NON-STATIONARY PULSE THROUGH A LAYER WITH DAMPING*

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The one-dimensional problem of the passage of a non-stationary stress pulse through an acoustic layer possessing internal friction is examined. The damping in the layer is described by the model of a Voigt medium /1/. The use of a Laplace transformation in time reduces the problem to the evaluation of a certain contour integral. The integrand has a denumerable number of poles and one essential singular point in the complex plane. It is proved that the integral under consideration can be evaluated in the form of a series of residues of the integrand.

1. Let a stress pulse $\sigma_z = p_0(t),\ z = 0$ be incident on a layer $0 \leqslant z \leqslant h$. We consider the layer to be a solid body possessing internal friction. The Voigt model /2/

$$\sigma_z = \lambda u' + \eta u'^{\cdot} \tag{1.1}$$

is the standard model for internal friction for acoustic wave propagation, where $u = u_z(z, t)$ is the displacement λ is the elastic modulus η is the viscosity, and differentiation with respect to time is denoted by a dot and with respect to the coordinate z by a prime. Adding the equation of motion $\rho u^{\cdot\cdot} = \sigma_z'$, to (1.1) we arrive at an equation in the function u

$$\rho u^{\cdot\cdot} = \lambda u'' + \eta u''^{\cdot} \tag{1.2}$$

To fix our ideas, we consider the opposite face of the layer stress-free. Then the boundary conditions have the form

$$\lambda u' + \eta u'^{\cdot} = \begin{cases} p_0(t), & z = 0 \\ 0, & z = h \end{cases} \tag{1.3}$$

For simplicity we consider the initial conditions to be zero $u = u^{\cdot} = 0,\ t = 0$.

Applying a Laplace transformation in time to the relationships (1.2) and (1.3), we obtain for the most interesting characteristic, namely, the rate of displacement of the face $z = h$

$$v(t, h) = \frac{1}{2\pi i\rho} \int\limits_{\delta - i\infty}^{\delta + i\infty} \frac{P_0(s)\, e^{st} ds}{\operatorname{sh}(hs/\gamma)},\quad \gamma = \gamma(s) = \sqrt{c^2 + \varepsilon s},\ \delta > 0 \tag{1.4}$$

$$c = \sqrt{\lambda/\rho},\quad \varepsilon = \eta/\rho$$

Here c is the speed of sound, and $P_0(s)$ is the Laplace transform of the function $p_0(t)$.

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